# On the area of the symmetry orbits in $T^2$ symmetric spacetimes with Vlasov matter

Marsha Weaver

Theoretical Physics Institute

University of Alberta

Edmonton, AB, Canada T6G 2J1

mweaver@phys.ualberta.ca

## Abstract

This paper treats the global existence question for a collection of general relativistic collisionless particles, all having the same mass. The spacetimes considered are globally hyperbolic, with Cauchy surface a three-torus. Furthermore, the spacetimes considered are isometrically invariant under a two-dimensional group action, the orbits of which are spacelike two-tori. It is known from previous work that the area of the group orbits serves as a global time coordinate. In the present work it is shown that the area takes on all positive values in the maximal Cauchy development.

## 1 Introduction

Global existence results have been obtained [1, 2, 3, 4, 5, 6, 7] for various classes of $T^2$ symmetric spacetimes – solutions to Einstein's equations with a closed (compact without boundary) Cauchy surface isometrically invariant under an effective group action, $U(1) \times U(1)$. A key feature of these results is that the area of the two-dimensional group orbits serves as a global time coordinate. In [1, 2, 6] the global existence result is obtained directly in terms of a coordinate system with areal time coordinate. In these cases the "areal coordinates" are also "conformal coordinates" and the area tending to

zero signals the end of the maximal Cauchy development.[1] In the other cases [3, 4, 5, 7], areal coordinates and conformal coordinates are distinct, and the global existence argument in the contracting direction is not made directly in terms of areal coordinates, but rather in terms of conformal coordinates. (The quotient of the spacetime by the symmetry group is conformally flat. In conformal coordinates the conformal factor is given explicitly.) The results in the second group leave open the possibility that any conformal time coordinate is unbounded in the contracting direction and that the area tends to some positive number (rather than all the way to zero) at the end of the maximal Cauchy development.

In [8], vacuum $T^2$ symmetric spacetimes such that the twist quantities associated with the two commuting spatial Killing vector fields do not both vanish are considered. Areal coordinates are used directly in the contracting direction to show that the area of the $T^2$ symmetry group orbits tends to zero at the end of the maximal Cauchy development (as long as the spacetime is not flat – one can choose two Killing vector fields tangent to a Cauchy surface in a spatially compactified flat Kasner spacetime such that the twist quantities are not both vanishing and such that the area of the $T^2$ group orbits tends to a positive number at the end of the maximal Cauchy development). The argument used in [8] to thus sharpen the vacuum global existence result previously obtained in [3] carries over straightforwardly to sharpen in the same way the global existence result for spacetimes with Gowdy symmetry (*i.e.,* vanishing twist quantities) and a nonvanishing magnetic field orthogonal to the group orbits obtained in appendix C of [5]. In the present work, the general strategy of [8], using areal coordinates directly in the contracting direction rather than conformal coordinates, is used to so sharpen global existence results previously obtained in [4, 7] for Einstein-Vlasov initial data on $T^3$ with $T^2$ symmetry.

The Einstein-Vlasov system models a general relativistic collection of collisionless particles. For an introduction to the Einstein-Vlasov system, see [9, 10, 11, 12]. In the present work, as in [4, 7], massive particles are considered, all having the same mass. In section 2, the areal coordinate metric is written out, a global existence result from [7] is recalled, and the Vlasov matter distribution function is introduced. Some preliminary results per-

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[1]The spatial topology is the three-torus in [1, 3, 4, 5, 6, 7] and a global time coordinate can be set equal to the area of the group orbits. In [2], other spatial topologies are considered, and the relationship of the area of the group orbits to the time coordinate is specified.

taining to the Vlasov matter, which are key to the presence of Vlasov matter insuring that the area of the $T^2$ symmetry orbits goes to zero, are obtained. In section 3, Einstein's equations are written out. In section 4, bounds on the areal metric functions are obtained in the contracting direction which are sufficient to allow application to the contracting direction of the techniques used in [3, 4, 7] for extending the spacetime in the expanding direction. These techniques involve integrating quantities related to the metric functions along the light cones of the quotient spacetime and integrating quantities related to the Vlasov matter along timelike geodesics. The main theorem appears in section 5.

There does exist an exceptional possibility for $T^2$ symmetric spacetimes, characterized by the spacetime gradient of the area of the symmetry group orbits vanishing on the closed Cauchy surface, which implies that the area of the orbits is everywhere constant, and so cannot serve as a time coordinate. In this exceptional case, the spacetime is flat, either $T^3 \times R$ Minkowski spacetime or flat Kasner. The presence of a nonvanishing twist quantity rules out this exceptional possibility [2], as does the presence of Vlasov matter (see the appendix).

## 2   Preliminaries

One of the results in [7] is that (except for the special vacuum solutions with symmetry group orbits of constant area mentioned in the introduction) the maximal Cauchy development of $T^2$ symmetric $C^\infty$ initial data on $T^3$ for the Einstein-Vlasov equations[2] is isometric to $(t_0, \infty) \times T^3$ with $C^\infty$ metric,

$$
\begin{aligned}
g \;=\; & e^{2(\nu-U)}(-\alpha\,dt^2 + d\theta^2) + e^{2U}[dx + A\,dy + (G + AH)\,d\theta]^2 \\
& + e^{-2U}t^2[dy + H\,d\theta]^2,
\end{aligned} \tag{1}
$$

for some nonnegative number $t_0$. All the metric functions are periodic in $\theta$ with period one and independent of $x$ and $y$ so that $X = \partial_x$ and $Y = \partial_y$ are commuting Killing vector fields. The area of the symmetry orbits is $t$. (The result in [4] is similar, but with an additional assumption made, that $G$ and $H$ vanish.)

The goal of the present work is to show that $t_0 = 0$, that the area of the group orbits goes to zero at the end of the maximal Cauchy development,

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[2]The area of the group orbits need not be constant on the initial data surface. The hypotheses concerning the Vlasov matter are stated later in this section.

in the contracting direction. The strategy for showing that $t_0 = 0$ will be to show that, given a $C^\infty$ solution to the Einstein-Vlasov equations with metric (1) on $(t_0, \infty) \times T^3$, and a nonvanishing number of Vlasov particles (see below), then for any $t_p \in [t_0, \infty)$ such that $t_p > 0$, the spacetime extends to $t_p$. It will be convenient to fix $t_p \in [t_0, \infty)$ such that $t_p > 0$, and to fix some arbitrary $t_i \in (t_p, \infty)$.

There are two twist quantities associated with the commuting Killing vector fields, $J = \epsilon_{\mu\rho\sigma\tau} X^\mu Y^\rho \nabla^\sigma X^\tau$ and $K = \epsilon_{\mu\rho\sigma\tau} X^\mu Y^\rho \nabla^\sigma Y^\tau$. If the orientation is such that $\{\partial_t, \partial_\theta, \partial_x, \partial_y\}$ is right handed, then these twist quantities are related to the metric functions as follows,[3]

$$J = -\frac{te^{-2\nu+4U}}{\sqrt{\alpha}}(G_t + AH_t) \quad \text{and} \quad K = AJ - \frac{t^3 e^{-2\nu}}{\sqrt{\alpha}}H_t. \tag{2}$$

Consider massive Vlasov particles all of the same mass. Without loss of generality, the mass can then be normalized to one. Let the velocity vector of the Vlasov matter be future directed (with the future designated as increasing $t$). Let the quantities, $v_\mu$, denote the components of the velocity vector in the frame

$$\{dt, \ d\theta, \ dx + G\, d\theta, \ dy + H\, d\theta\}. \tag{3}$$

With this choice of frame, $v_2$ and $v_3$ are constant along geodesics, and from $v_\mu v^\mu = -1$,

$$v_0 = -\sqrt{\alpha e^{2\nu-2U} + \alpha v_1^2 + \alpha e^{2\nu-4U} v_2^2 + \alpha t^{-2} e^{2\nu}(v_3 - Av_2)^2}. \tag{4}$$

The Vlasov matter distribution function, $f$, is in general a nonnegative function of position and velocity (or momentum). Suppose that $C^\infty$ initial data is given for the Einstein-Vlasov equations[4] on some $T^2$ symmetric initial three-torus, with a nonvanishing number of Vlasov particles (see the paragraph following lemma 2 below), with the matter distribution function invariant under the action of the symmetry group and with the support of $f$ bounded initially. From [7] it is known that $f$ is $C^\infty$ on $(t_0, \infty) \times T^3 \times R^3$, and the support of $f$ is bounded at each $t \in (t_0, \infty)$. The Vlasov equation is that $f$ should be constant along timelike geodesics. In coordinates $(t, \theta, x, y, v_1, v_2, v_3)$, the Vlasov equation takes the form

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[3]With this choice of orientation, $K$ as defined here is related to $K$ as defined in [3] and [8] by change of sign. $J = 0$ in [3] and [8].

[4]See [10, 11, 12] for statements of the Einstein-Vlasov initial value problem.

$$\frac{\partial f}{\partial t} = \frac{\partial v_0}{\partial v_1} \frac{\partial f}{\partial \theta} - \left\{ \frac{\partial v_0}{\partial \theta} \right.$$
$$\left. + \frac{\sqrt{\alpha} e^{2\nu}}{t^3} (K - A\,J)(v_3 - A\,v_2) + \frac{\sqrt{\alpha} e^{2\nu - 4U}}{t} J v_2 \right\} \frac{\partial f}{\partial v_1}, \quad (5)$$

with $f_x = 0$, $f_y = 0$ and $f$ periodic in $\theta$ with period one.

**Lemma 1** *There exists a number $C_1 > 0$ such that $f \leq C_1$ on $(t_p, t_i] \times T^3 \times R^3$.*

*Proof:* This follows from $f$ being constant on timelike geodesics. $\square$

The second lemma follows from the continuity equation, $\nabla_\mu N^\mu = 0$, which holds generally for Vlasov matter, with $N^\mu$ the matter current density. In the frame (3), the matter current density is

$$N^\mu = \frac{\sqrt{\alpha}}{t} \int_{R^3} \frac{f v^\mu}{|v_0|} \, dv_1 \, dv_2 \, dv_3.$$

**Lemma 2** *The quantity $C_2 = \int_{S^1} \left( \int_{R^3} f \, dv_1 \, dv_2 \, dv_3 \right) d\theta$ is constant in time.*

*Proof:* This lemma can be obtained directly from the continuity equation. Similarly, it can be obtained from equation (5), as follows.

$$\frac{dC_2}{dt} = \int_{S^1} \left( \int_{R^3} \left\{ \frac{\partial v_0}{\partial v_1} \frac{\partial f}{\partial \theta} - \left[ \frac{\partial v_0}{\partial \theta} + \frac{\sqrt{\alpha} e^{2\nu}}{t^3} (K - A\,J)(v_3 - A\,v_2) \right. \right. \right.$$
$$\left. \left. \left. + \frac{\sqrt{\alpha} e^{2\nu - 4U}}{t} J v_2 \right] \frac{\partial f}{\partial v_1} \right\} dv_1 \, dv_2 \, dv_3 \right) d\theta, \quad (6)$$

$$= \int_{S^1} \left( \int_{R^3} \left\{ \frac{\partial v_0}{\partial v_1} \frac{\partial f}{\partial \theta} + \frac{\partial^2 v_0}{\partial v_1 \partial \theta} f \right\} dv_1 \, dv_2 \, dv_3 \right) d\theta, \quad (7)$$

$$= \int_{S^1} \left( \int_{R^3} \frac{\partial}{\partial \theta} \left( \frac{\partial v_0}{\partial v_1} f \right) dv_1 \, dv_2 \, dv_3 \right) d\theta,$$

$$= 0. \quad (8)$$

Integration by parts leads from line (6) to line (7). $\square$

The property that the number of Vlasov particles is nonvanishing is equivalent to $C_2 \neq 0$, which is assumed throughout the rest of this work.

Since $f$ doesn't grow arbitrarily large, since the integral of $f$ is a constant in time, and since $f$, $v_2$ and $v_3$ are all constant along timelike geodesics, $\int_{S^1} \left( \int_{R^3} f|v_1| \, dv_1 \, dv_2 \, dv_3 \right) d\theta$ does not become arbitrarily small on $(t_p, t_i]$, as shown by the following lemma.

**Lemma 3** *There exists $C_3 > 0$ such that $\int_{S^1} \left( \int_{R^3} f|v_1|\ dv_1\, dv_2\, dv_3 \right) d\theta > C_3$ on $(t_p, t_i]$.*

*Proof:* Fix positive numbers $\bar{v}_2$ and $\bar{v}_3$ such that the support of $f$ at $t_i$ is contained in $S^1 \times R^1 \times [-\bar{v}_2, \bar{v}_2] \times [-\bar{v}_3, \bar{v}_3]$. At any $t \in (t_p, t_i]$, the support of $f$ is contained within this same set. This is because $f$, $v_2$ and $v_3$ are constant along timelike geodesics, so that if $f$ vanishes for all $\theta$ and $v_1$ at some $(t_i, v_2, v_3)$ then $f$ vanishes for all $\theta$ and $v_1$ at $(t, v_2, v_3)$ for all $t \in (t_p, t_i]$. Fix $\delta > 0$ small enough so that $b = C_2 - 8\delta\bar{v}_2\bar{v}_3 C_1$ is positive. From the preceding paragraph and lemma 1,

$$\int_{S^1} \left( \int_{R^2} \left( \int_{-\delta}^{\delta} f\, dv_1 \right) dv_2\, dv_3 \right) d\theta \le 8\delta\bar{v}_2\bar{v}_3 C_1. \tag{9}$$

Now using lemma 2,

$$\int_{S^1} \left( \int_{R^2} \left( \int_{|v_1|>\delta} f\, dv_1 \right) dv_2\, dv_3 \right) d\theta \ge b.$$

To obtain lemma 3 calculate

$$
\begin{aligned}
\int_{S^1} \left( \int_{R^3} f|v_1|\, dv_1\, dv_2\, dv_3 \right) d\theta &= \int_{S^1} \left( \int_{R^2} \left( \int_{-\delta}^{\delta} f|v_1|\, dv_1 \right) dv_2\, dv_3 \right) d\theta \\
&\quad + \int_{S^1} \left( \int_{R^2} \left( \int_{|v_1|>\delta} f|v_1|\, dv_1 \right) dv_2\, dv_3 \right) d\theta \\
&\ge \delta \int_{S^1} \left( \int_{R^2} \left( \int_{|v_1|>\delta} f\, dv_1 \right) dv_2\, dv_3 \right) d\theta \\
&\ge \delta b, \tag{10}
\end{aligned}
$$

and let $C_3 = \delta b$. $\qquad\square$

In the course of the proof of the main result, a coordinate transformation, $(t, \theta, v_1, v_2, v_3) \to (s, \phi, w_1, w_2, w_3)$, will be used, with $t = s$ (letting $s_i = t_i$ and $s_p = t_p$), $v_2 = w_2$, $v_3 = w_3$, $\partial\theta/\partial s = -\partial v_0/\partial v_1$,

$$\frac{\partial v_1}{\partial s} = \frac{\partial v_0}{\partial \theta} + \sqrt{\alpha}e^{2\nu}\left\{ \frac{(K - A\,J)(v_3 - A\,v_2)}{t^3} + \frac{e^{-4U}J\,v_2}{t} \right\},$$

$\theta(s_i, \phi, w_1, w_2, w_3) = \phi$ and $v_1(s_i, \phi, w_1, w_2, w_3) = w_1$. The new coordinates, $(\phi, w_1, w_2, w_3)$, are constant on timelike geodesics and therefore $f_s = 0$, which is why the coordinate transformation is useful.

**Lemma 4** *The Jacobian determinant of the transformation* $(t, \theta, v_1, v_2, v_3) \rightarrow$ $(s, \phi, w_1, w_2, w_3)$, *equals one on* $(s_p, s_i] \times S^1 \times R^3$.

*Proof:* The Jacobian determinant of the coordinate transformation is

$$\eta = \frac{\partial \theta}{\partial \phi} \frac{\partial v_1}{\partial w_1} - \frac{\partial \theta}{\partial w_1} \frac{\partial v_1}{\partial \phi}.$$

On $\{s_i\} \times S^1 \times R^3$, $\eta = 1$, and on $(s_p, s_i] \times S^1 \times R^3$,

$$
\begin{aligned}
\eta_s \; = \; & \frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial s}\right) \frac{\partial \theta}{\partial \phi} \frac{\partial v_1}{\partial w_1} + \frac{\partial}{\partial v_1}\left(\frac{\partial \theta}{\partial s}\right) \frac{\partial v_1}{\partial \phi} \frac{\partial v_1}{\partial w_1} \\
& + \frac{\partial \theta}{\partial \phi} \frac{\partial}{\partial \theta}\left(\frac{\partial v_1}{\partial s}\right) \frac{\partial \theta}{\partial w_1} + \frac{\partial \theta}{\partial \phi} \frac{\partial}{\partial v_1}\left(\frac{\partial v_1}{\partial s}\right) \frac{\partial v_1}{\partial w_1} \\
& - \frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial s}\right) \frac{\partial \theta}{\partial w_1} \frac{\partial v_1}{\partial \phi} - \frac{\partial}{\partial v_1}\left(\frac{\partial \theta}{\partial s}\right) \frac{\partial v_1}{\partial w_1} \frac{\partial v_1}{\partial \phi} \\
& - \frac{\partial \theta}{\partial w_1} \frac{\partial}{\partial \theta}\left(\frac{\partial v_1}{\partial s}\right) \frac{\partial \theta}{\partial \phi} - \frac{\partial \theta}{\partial w_1} \frac{\partial}{\partial v_1}\left(\frac{\partial v_1}{\partial s}\right) \frac{\partial v_1}{\partial \phi}.
\end{aligned}
\tag{11}
$$

From

$$\frac{\partial}{\partial \theta}\left(\frac{\partial \theta}{\partial s}\right) + \frac{\partial}{\partial v_1}\left(\frac{\partial v_1}{\partial s}\right) = 0,$$

and cancellation of terms in (11), it follows that $\eta_s = 0$. $\qquad\square$

In general, coordinates that are constant on timelike geodesics are valid on the maximal Cauchy development of an initial data surface. Here one can see this also directly from lemma 4 combined with the smoothness of the metric functions appearing in (1) on the maximal Cauchy devolopment.

## 3 Einstein's equations

With choice of frame (3), the stress-energy tensor due to the Vlasov matter is

$$T_{\mu\rho} = \frac{\sqrt{\alpha}}{t} \int_{R^3} \frac{f v_\mu v_\rho}{|v_0|} dv_1 \, dv_2 \, dv_3.
\tag{12}$$

Einstein's equations, $G_{\mu\rho} = T_{\mu\rho}$, for the metric (1) with source (12) are satisfied if

$$
\nu_t = t\left\{U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2}(A_t^2 + \alpha A_\theta^2)\right\} + \frac{\alpha e^{2\nu-4U}}{4t}J^2
$$

$$
+ \frac{\alpha e^{2\nu}}{4t^3}(K - A\,J)^2 + \sqrt{\alpha}\int_{R^3} f|v_0|\,dv_1\,dv_2\,dv_3, \tag{13}
$$

$$
\nu_\theta = 2t\left(U_tU_\theta + \frac{e^{4U}}{4t^2}A_tA_\theta\right) - \frac{\alpha_\theta}{2\alpha} - \sqrt{\alpha}\int_{R^3} f\,v_1\,dv_1\,dv_2\,dv_3, \tag{14}
$$

$$
\frac{\alpha_t}{\alpha} = -\frac{\alpha e^{2\nu-4U}J^2}{t} - \frac{\alpha e^{2\nu}(K - A\,J)^2}{t^3}
$$

$$
-2\alpha^{3/2}e^{2\nu}\int_{R^3} \frac{f(e^{-2U} + e^{-4U}v_2^2 + t^{-2}(v_3 - Av_2)^2)}{|v_0|}dv_1\,dv_2\,dv_3, \tag{15}
$$

$$
J_t = -2\alpha\int_{R^3}\frac{fv_1v_2}{|v_0|}dv_1\,dv_2\,dv_3, \tag{16}
$$

$$
J_\theta = 2\int_{R^3} fv_2\,dv_1\,dv_2\,dv_3, \tag{17}
$$

$$
K_t = -2\alpha\int_{R^3}\frac{fv_1v_3}{|v_0|}dv_1\,dv_2\,dv_3, \tag{18}
$$

$$
K_\theta = 2\int_{R^3} fv_3\,dv_1\,dv_2\,dv_3, \tag{19}
$$

$$
U_{tt} = \alpha U_{\theta\theta} - \frac{U_t}{t} + \frac{\alpha_tU_t}{2\alpha} + \frac{\alpha_\theta U_\theta}{2} + \frac{e^{4U}}{2t^2}\left(A_t{}^2 - \alpha A_\theta{}^2\right)
$$

$$
+\frac{\alpha e^{2\nu-4U}}{2t^2}J^2 + \frac{\alpha^{3/2}e^{2\nu-2U}}{2t}\int_{R^3}\frac{f(1 + 2e^{-2U}v_2^2)}{|v_0|}dv_1\,dv_2\,dv_3, \tag{20}
$$

$$
A_{tt} = \alpha A_{\theta\theta} + \frac{A_t}{t} + \frac{\alpha_tA_t}{2\alpha} + \frac{\alpha_\theta A_\theta}{2} - 4(A_tU_t - \alpha A_\theta U_\theta)
$$

$$
+\frac{\alpha e^{2\nu-4U}}{t^2}J(K - A\,J)
$$

$$
+\frac{2\alpha^{3/2}e^{2\nu-4U}}{t}\int_{R^3}\frac{fv_2(v_3 - Av_2)}{|v_0|}dv_1\,dv_2\,dv_3. \tag{21}
$$

Since $J$ and $K$ are periodic in $\theta$, equations (17) and (19) impose requirements on $f$. Namely, $\int_{S^1}\int_{R^3} fv_2\,dv_1\,dv_2\,dv_3\,d\theta = 0$ and $\int_{S^1}\int_{R^3} fv_3\,dv_1\,dv_2\,dv_3\,d\theta = 0$. Similarly, equation (14) imposes the condition

$$
\int_{S^1}\left\{2t\left(U_tU_\theta + \frac{e^{4U}}{4t^2}A_tA_\theta\right) - \sqrt{\alpha}\int_{R^3} f\,v_1\,dv_1\,dv_2\,dv_3\right\}d\theta = 0. \tag{22}
$$

All three of these conditions are preserved by the evolution. That this is so is equivalent to the integrability conditions for equations (16) and (17), equations (18) and (19), and equations (13) and (14), respectively.

It is useful to also have at hand an equation which can be derived from (5) and (13) - (21) or, alternatively, can be obtained directly from Einstein's equations, $G_{\mu\rho} = T_{\mu\rho}$,

$$
\begin{aligned}
\nu_{tt} &= \alpha\nu_{\theta\theta} + \frac{\alpha_t\nu_t}{2\alpha} + \frac{\alpha_\theta\nu_\theta}{2} + \frac{\alpha_{\theta\theta}}{2} - \frac{\alpha_\theta^2}{4\alpha} - U_t^2 + \alpha U_\theta^2 \\
&\quad + \frac{e^{4U}}{4t^2}\left(A_t{}^2 - \alpha A_\theta{}^2\right) - \frac{\alpha e^{2\nu-4U}J^2}{4t^2} - \frac{3\alpha e^{2\nu}(K - A\,J)^2}{4t^4} \\
&\quad - \frac{\alpha^{3/2}e^{2\nu}}{t^3}\int_{R^3} \frac{f(v_3 - A\,v_2)^2}{|v_0|}\,dv_1\,dv_2\,dv_3.
\end{aligned} \tag{23}
$$

# 4    Extending the spacetime

Next a number of lemmas are presented which give sufficient control of the metric functions in the contracting direction so that, later in this section, a slight modification of the combination light cone/timelike geodesic arguments used in [4, 7] for the expanding direction can be applied to the contracting direction. These arguments give control of $U$ and $A$ and their first derivatives and also the support of $f$ (proposition 1), followed by control of the first derivatives of $f$ (proposition 2). $C^\infty$ bounds on all the metric functions and on $f$ follow from light cone/timelike geodesic arguments and the Einstein-Vlasov equations.

**Lemma 5** *For any number $b$, $\sqrt{\alpha}e^{2\nu+bU}$ is bounded on $(t_p, t_i] \times S^1$.*

*Proof:*

$$
\begin{aligned}
\partial_t(t^{b^2/8}\sqrt{\alpha}e^{2\nu+bU}) &= (t^{b^2/8}\sqrt{\alpha}e^{2\nu+bU})\Big\{2t\Big[(U_t + \frac{b}{4t})^2 \\
&\quad + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2}(A_t^2 + \alpha A_\theta^2)\Big] \\
&\quad + \sqrt{\alpha}\int_{R^3} f\left(|v_0| + \frac{\alpha v_1^2}{|v_0|}\right)dv_1\,dv_2\,dv_3\Big\}, \tag{24} \\
&\geq 0. \tag{25}
\end{aligned}
$$

So on $(t_p, t_i] \times S^1$,

$$\sqrt{\alpha(t,\theta)}e^{2\nu(t,\theta)+bU(t,\theta)} \leq \frac{t_i^{b^2/8}}{t_p^{b^2/8}}\sqrt{\alpha(t_i,\theta)}e^{2\nu(t_i,\theta)+bU(t_i,\theta)}.$$

$\square$

**Lemma 6** *For any positive number $r$ and any number $\lambda$,*

$$\alpha^{r/2}e^{2r\nu+\lambda U}A^2$$

*is bounded on $(t_p, t_i] \times S^1$.*

*Proof:* An $SL(2,R)$ transformation of the Killing vectors relates the quantity under consideration here to quantities of the form considered in lemma 5.

Consider tilded coordinates, defined by $\partial_{\tilde{t}} = \partial_t$, $\partial_{\tilde{\theta}} = \partial_\theta$,

$$\tilde{X} = \partial_{\tilde{x}} = aX + bY,$$

and

$$\tilde{Y} = \partial_{\tilde{y}} = cX + dY,$$

with $a$, $b$, $c$ and $d$ constants such that $ad - bc = 1$. Then the form of the metric is unchanged from (1), from which it follows that the tilded metric functions satisfy "tilded" equations (13) - (23). The relation between the tilded metric functions and the untilded metric functions is

$$e^{2\tilde{U}} = e^{2U}(a + Ab)^2 + t^2 e^{-2U}b^2 \tag{26}$$
$$e^{2\tilde{U}}\tilde{A} = e^{2U}(a + Ab)(c + Ad) + t^2 e^{-2U}bd \tag{27}$$
$$\tilde{\alpha} = \alpha \tag{28}$$
$$\tilde{\alpha}e^{2\tilde{\nu}-2\tilde{U}} = \alpha e^{2\nu-2U} \tag{29}$$

Note that $\tilde{t} = t$ and $\tilde{\theta} = \theta$ and now consider $a = 0$ and $b = 1$. Choose a positive number $q < r$. From equations (26), (28) and (29),

$$\tilde{\alpha}^{q/2}e^{2q\tilde{\nu}+2(1-q)\tilde{U}} = \alpha^{q/2}e^{2q\nu+2(1-q)U}A^2 + \alpha^{q/2}t^2 e^{2q\nu-2(1+q)U}. \tag{30}$$

Since the tilded metric functions satisfy the same equations as the untilded metric functions, the left hand side of equation (30) is bounded on $(t_p, t_i] \times S^1$

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from lemma 5. Since the second term on the right hand side of equation (30) is positive, $\alpha^{q/2}e^{2q\nu+2(1-q)U}A^2$ is bounded on $(t_p, t_i] \times S^1$. Since $\alpha^{(r-q)/2}e^{2(r-q)\nu+(\lambda-2+2q)U}$ is bounded on $(t_p, t_i] \times S^1$ from lemma 5, lemma 6 follows, by taking the product of these two bounded quantities. $\square$

Next the monotonic quantity from [7] will be recalled. In [7] this quantity was called $E(t)$. The notation here follows [8]. Let

$$\tilde{\mathcal{E}}(t) = \int_{S^1} \frac{\nu_t}{\sqrt{\alpha}t} \, d\theta. \tag{31}$$

**Lemma 7** $d\tilde{\mathcal{E}}/dt < 0$.

*Proof:*

$$
\begin{aligned}
\frac{d\tilde{\mathcal{E}}}{dt} &= -\int_{S^1} \left\{ \frac{2}{t} \left( \frac{U_t^2}{\sqrt{\alpha}} + \frac{e^{4U}}{4t^2}\sqrt{\alpha}A_\theta^2 \right) + \frac{\sqrt{\alpha}e^{2\nu-4U}}{2t^3}J^2 + \frac{\sqrt{\alpha}e^{2\nu}}{t^5}(K - A\,J)^2 \right. \\
&\quad \left. + \int_{R^3} \left( \frac{f|v_0|}{t^2} + \frac{\alpha e^{2\nu} f(v_3 - A\,v_2)^2}{t^4|v_0|} \right) dv_1 \, dv_2 \, dv_3 \right\} d\theta. \tag{32}
\end{aligned}
$$

Equation (32) follows from the evolution equations (15) and (23) and from the vanishing of the integral over the circle of the derivative with respect to $\theta$ of any $C^1$ function on the circle. A nonvanishing number of Vlasov particles ($C_2 \neq 0$) insures that $d\tilde{\mathcal{E}}/dt$ is strictly negative. $\square$

**Lemma 8** $\tilde{\mathcal{E}}(t)$ *is bounded on* $(t_p, t_i]$. *There exists a number,* $\tilde{\mathcal{E}}_p$, *satisfying*

$$\tilde{\mathcal{E}}_p = \lim_{t \downarrow t_p} \tilde{\mathcal{E}}(t).$$

*Proof:* From equation (32), $d\tilde{\mathcal{E}}/dt \geq -4\tilde{\mathcal{E}}/t$. So for any $t_k \in (t_p, t_i]$,

$$\tilde{\mathcal{E}}(t_k) \leq \tilde{\mathcal{E}}(t_i) + \int_{t_k}^{t_i} \frac{4\tilde{\mathcal{E}}(t)}{t} \, dt. \tag{33}$$

Applying Gronwall's lemma to this inequality (as suggested on p 353 of [4]), we obtain

$$\tilde{\mathcal{E}}(t_k) \leq \tilde{\mathcal{E}}(t_i) \left( \frac{t_i}{t_k} \right)^4, \tag{34}$$

for any $t_k \in (t_p, t_i]$. Thus $\tilde{\mathcal{E}}(t_k) < \tilde{\mathcal{E}}(t_i)(t_i/t_p)^4$ on $(t_p, t_i)$. This bound, together with the monotonicity of $\tilde{\mathcal{E}}(t)$, guarantees that $\tilde{\mathcal{E}}_p$, as defined in the statement of the lemma, exists. $\square$

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Let
$$\beta = \nu + \frac{\ln \alpha}{2}. \tag{35}$$

The substitution of $e^\beta$ for $\sqrt{\alpha}e^\nu$ will be made freely throughout the rest of this section. This is a convenience, because of the order in which bounds are obtained. The advantage of $\beta$ over $\nu$ is that $\alpha_\theta$ does not appear in the expression for $\beta_\theta$ obtained from equation (14),

$$\beta_\theta = 2t \left( U_t U_\theta + \frac{e^{4U}}{4t^2} A_t A_\theta \right) - \sqrt{\alpha} \int_{R^3} f \, v_1 \, dv_1 \, dv_2 \, dv_3. \tag{36}$$

**Lemma 9** *There exists a positive number, $C_4$, such that $\int_{S^1} |\beta_\theta| \, d\theta$, $\int_{S^1} |U_\theta| \, d\theta$, $\int_{S^1} e^{2U} |A_\theta| \, d\theta$, $\int_{S^1} |J_\theta| d\theta$ and $\int_{S^1} |K_\theta| d\theta$ are each less than $C_4$ on $(t_p, t_i]$. Thus, the difference between the spatial maximum and minimum values of each of the quantities, $\beta$, $U$, $J$ and $K$ on $S^1$ (at any fixed time $t$) is less than $C_4$.*

*Proof:* Using that $\alpha$ and $\tilde{\mathcal{E}}$ are increasing with decreasing $t$,

$$\int_{S^1} |\beta_\theta| \, d\theta \;\leq\; t_i \tilde{\mathcal{E}}_p \tag{37}$$

$$\int_{S^1} |U_\theta| \, d\theta \;\leq\; \frac{\sqrt{\tilde{\mathcal{E}}_p}}{(\min_{S_1} \alpha(t_i, \theta))^{1/4}}, \tag{38}$$

$$\int_{S^1} e^{2U} |A_\theta| \, d\theta \;\leq\; \frac{2t_i \sqrt{\tilde{\mathcal{E}}_p}}{(\min_{S_1} \alpha(t_i, \theta))^{1/4}}, \tag{39}$$

$$\int_{S^1} |J_\theta| d\theta \;\leq\; 2\bar{v}_2 C_2, \tag{40}$$

$$\int_{S^1} |K_\theta| d\theta \;\leq\; 2\bar{v}_3 C_2, \tag{41}$$

on $(t_p, t_i]$. $\qquad \square$

**Lemma 10** $\min_{S_1} \alpha(t, \theta)$ *is bounded on $(t_p, t_i]$.*

*Proof:* For any $t \in (t_p, t_i]$, (see equations (31) and (13)),

$$\int_{S^1} \int_{R^3} f|v_0| \, dv_1 \, dv_2 \, dv_3 \, d\theta \leq t \tilde{\mathcal{E}}(t). \tag{42}$$

Now considering equation (4),

$$\int_{S^1} \sqrt{\alpha} \int_{R^3} f|v_1| \, dv_1 \, dv_2 \, dv_3 \, d\theta \leq t \tilde{\mathcal{E}}(t).$$

Therefore,

$$\sqrt{\min_{S_1} \alpha(t,\theta)} \int_{S^1} \int_{R^3} f|v_1| \, dv_1 \, dv_2 \, dv_3 \, d\theta \leq t\tilde{\mathcal{E}}(t).$$

From lemma 3,

$$\sqrt{\min_{S_1} \alpha(t,\theta)} C_3 \leq t\tilde{\mathcal{E}}(t),$$

with $C_3 > 0$. So

$$\sqrt{\min_{S_1} \alpha(t,\theta)} \leq \frac{t\tilde{\mathcal{E}}(t)}{C_3}.$$

From lemmas 7 and 8,

$$\min_{S_1} \alpha(t,\theta) \leq \left( \frac{t_i \tilde{\mathcal{E}}_p}{C_3} \right)^2 \tag{43}$$

on $(t_p, t_i]$. $\qquad\square$

**Lemma 11** *There exists $\bar{\theta} \in S_1$ such that $\alpha(t,\bar{\theta})$ is bounded on $(t_p, t_i]$.*

*Proof:* Suppose there existed no $\bar{\theta}$ such that $\alpha(t,\bar{\theta})$ were bounded on $(t_p, t_i]$. Since $\alpha(t,\theta)$ is increasing with decreasing $t$, this would contradict the previous lemma. $\qquad\square$

**Corollary 1** *On $(t_p, t_i]$,*

$$\int_t^{t_i} \frac{\alpha_\sigma(\sigma,\bar{\theta})}{\alpha(\sigma,\bar{\theta})} \, d\sigma$$

*is bounded.*

*Proof:* $\ln \alpha(t,\bar{\theta})$ is bounded since $\alpha(t,\bar{\theta})$ is bounded and increasing with decreasing $t$. $\qquad\square$

In the next four lemmas control of the metric functions is obtained which is sufficient for using the light cone/timelike geodesic arguments directly in terms of areal coordinates in the contracting direction.

**Lemma 12** *For any number $b$, $e^{\beta+bU}$ is bounded on $(t_p, t_i] \times S^1$.*

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*Proof:* It follows from lemma 5 along with $\alpha$ being bounded on $(t_p, t_i] \times \{\bar{\theta}\}$ that $e^{\beta + bU}$ is bounded on $(t_p, t_i] \times \{\bar{\theta}\}$. (See equation (35) for the definition of $\beta$.) That $e^{\beta + bU}$ is bounded on $(t_p, t_i] \times S^1$ then follows from the boundedness of the difference between the maximum and minimum values of $\beta$ and $U$ on $S^1$ (see lemma 9). $\qquad\square$

**Lemma 13** *For any positive number $r$ and any number $b$,*

$$e^{r\beta + bU} A$$

*is bounded on $(t_p, t_i] \times S^1$.*

*Proof:* It follows from lemma 6 along with $\alpha$ being bounded on $(t_p, t_i] \times \{\bar{\theta}\}$ that $e^{r\beta + bU} A$ is bounded on $(t_p, t_i] \times \{\bar{\theta}\}$. Integrating $(e^{r\beta + bU} A)_\theta$ along a $t =$ constant path from $(t, \bar{\theta})$ to $(t, \theta_2)$ leads to

$$
\begin{aligned}
(e^{r\beta + bU} A)(t, \theta_2) &= (e^{r\beta + bU} A)(t, \bar{\theta}) \\
&\quad + \int_{\bar{\theta}}^{\theta_2} \left\{ e^{r\beta + bU} A(r\beta_\theta + bU_\theta) \right. \\
&\quad \left. + e^{r\beta + (b-2)U} e^{2U} A_\theta \right\} d\theta, \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
|(e^{r\beta + bU} A)(t, \theta_2)| &\leq |(e^{r\beta + bU} A)(t, \bar{\theta})| \\
&\quad + \left| \int_{\bar{\theta}}^{\theta_2} \left\{ e^{r\beta + bU} |A|(r|\beta_\theta| + |bU_\theta|) \right. \right. \\
&\quad \left. \left. + e^{r\beta + (b-2)U} e^{2U} |A_\theta| \right\} d\theta \right| \qquad (45)
\end{aligned}
$$

$$
\leq C_5 + \left| \int_{\bar{\theta}}^{\theta_2} e^{r\beta + bU} |A|(r|\beta_\theta| + |bU_\theta|)\, d\theta \right|, \qquad (46)
$$

on $(t_p, t_i]$, for some number $C_5 > 0$ (see lemmas 12 and 9). From Gronwall's lemma,

$$|(e^{r\beta + bU} A)(t, \theta_2)| \leq C_5 \exp \left| \int_{\bar{\theta}}^{\theta_2} (r|\beta_\theta| + |bU_\theta|)\, d\theta \right|, \qquad (47)$$

which (using lemma 9) determines a bound for $e^{r\beta + bU} A$ on $(t_p, t_i] \times S^1$. $\quad\square$

**Lemma 14**
$$\int_t^{t_i} \max_{S^1} [e^{2\beta(\sigma, \theta) - 4U(\sigma, \theta)} (J(\sigma, \theta))^2]\, d\sigma$$

*is bounded on $(t_p, t_i]$.*

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*Proof:* From lemma 9,

$$\int_t^{t_i} \max_{S^1}[e^{2\beta(\sigma,\theta)-4U(\sigma,\theta)} J(\sigma,\theta))^2]\, d\sigma$$

$$\leq \int_t^{t_i} e^{6C_4} e^{2\beta(\sigma,\bar{\theta})-4U(\sigma,\bar{\theta})}(J(\sigma,\bar{\theta})^2 + 2C_4|J(\sigma,\bar{\theta})| + C_4^2)d\sigma. \qquad (48)$$

The right hand side of inequality (48) is bounded from corollary 1, the evolution equation (15) and lemma 12. $\qquad\square$

**Lemma 15**

$$\int_t^{t_i} \max_{S^1}[e^{2\beta(\sigma,\theta)}(K(\sigma,\theta) - A(\sigma,\theta)\, J(\sigma,\theta))^2]d\sigma$$

*is bounded on* $(t_p, t_i]$.

*Proof:* Integrating $(e^{\beta}(K - A\,J))_\theta$ along a $t = $ constant path from $(t,\bar{\theta})$ to $(t,\theta_2)$ leads to

$$
\begin{aligned}
(e^{\beta}|K - A\,J|)(t,\theta_2) \;\leq\; & (e^{\beta}|K - A\,J|)(t,\bar{\theta}) + \left|\int_{\bar{\theta}}^{\theta_2}\left\{ e^{\beta}|K - A\,J||\beta_\theta|\right.\right. \\
& +e^{\beta}\int_{R^3} f|v_2|dv_1 dv_2 dv_3 \\
& +e^{\beta}A\int_{R^3} f|v_3|dv_1 dv_2 dv_3 \\
& \left.\left.+e^{\beta-2U}|J|e^{2U}|A_\theta|\right\}d\theta\right| \qquad (49) \\
\leq\; & (e^{\beta}|K - A\,J|)(t,\bar{\theta}) + C_6 \\
& +C_4 \max_{S^1}(e^{\beta-2U}|J|) \\
& +\left|\int_{\bar{\theta}}^{\theta_2} e^{\beta}|K - A\,J||\beta_\theta|d\theta\right|, \qquad (50)
\end{aligned}
$$

for any $(t,\theta_2) \in (t_p, t_i] \times S^1$, with

$$C_6 = C_2(\bar{v}_2 \sup_{(t_p,t_i]\times S^1} (e^{\beta}) + \bar{v}_3 \sup_{(t_p,t_i]\times S^1} (e^{\beta}A)).$$

From Gronwall's lemma,

$$
\begin{aligned}
(e^{\beta}|K - A\,J|)(t,\theta_2) \;\leq\; & e^{\left|\int_{\bar{\theta}}^{\theta_2}|\beta_\theta|\,d\theta\right|}\left\{(e^{\beta}|K - A\,J|)(t,\bar{\theta}) + C_6\right. \\
& \left.+C_4 \max_{S^1}(e^{\beta-2U}|J|)\right\}. \qquad (51)
\end{aligned}
$$

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Using lemma 9,

$$
\max_{S^1}[(e^\beta|K - A\,J|)(t,\theta)] \;\leq\; e^{C_4}\Big\{(e^\beta|K - A\,J|)(t,\bar\theta) + C_6
$$
$$
+ C_4 \max_{S^1}(e^{\beta-2U}|J|)\Big\}. \tag{52}
$$

Therefore,

$$
\int_t^{t_i} \max_{S^1}[e^{2\beta(\sigma,\theta)}(K(\sigma,\theta) - A(\sigma,\theta)\,J(\sigma,\theta))^2]\,d\sigma \;\leq\;
$$
$$
e^{2C_4}\int_t^{t_i}\Big\{e^{2\beta(\sigma,\bar\theta)}(K(\sigma,\bar\theta) - A(\sigma,\bar\theta)\,J(\sigma,\bar\theta))^2
$$
$$
+ 2C_6(e^\beta|K - A\,J|)(\sigma,\bar\theta)
$$
$$
+ 2(e^\beta|K - A\,J|)(\sigma,\bar\theta)C_4 \max_{S^1}(e^{\beta-2U}|J|)
$$
$$
+ \Big(C_6 + C_4 \max_{S^1}(e^{\beta-2U}|J|)\Big)^2 d\sigma\Big\}. \tag{53}
$$

Using that

$$
2(e^\beta|K - A\,J|)(\sigma,\bar\theta)C_4 \max_{S^1}(e^{\beta-2U}|J|)
$$
$$
\leq (e^{2\beta}(K - A\,J)^2)(\sigma,\bar\theta) + C_4^2 \max_{S^1}(e^{2\beta-4U}J^2),
$$

the right hand side of inequality (53) is bounded from corollary 1, the evolution equation (15) and lemma 14. □

**Proposition 1** *The functions $U$ and $A$ and their first derivatives are bounded on $(t_p, t_i] \times S^1$. The support of $f$ is bounded on $(t_p, t_i]$.*

*Proof:* The proof of this proposition (which has to do with the contracting direction) is a slight modification of the argument used for the expanding direction in [4] and [7]. Let

$$
E \;=\; \frac{1}{2}\{U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2}(A_t^2 + \alpha A_\theta^2)\}, \tag{54}
$$

$$
P \;=\; \sqrt{\alpha}(U_t U_\theta + \frac{e^{4U}}{4t^2}A_t A_\theta). \tag{55}
$$

Since $E \pm P$ are both the sum of squares, $|P| \leq E$. Set

$$
(\partial_t \mp \sqrt{\alpha}\partial_\theta)(E \pm P) = L_\pm. \tag{56}
$$

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A straightforward calculation shows that

$$
\begin{aligned}
L_{\pm} &= -\frac{1}{t}\left(U_t^2 + \frac{e^{4U}}{4t^2}\alpha A_\theta^2\right) + \frac{\alpha_t}{\alpha}(E \pm P) \mp \frac{P}{t} \\
&\quad + (U_t \pm \sqrt{\alpha}U_\theta)\left\{\frac{e^{2\beta-4U}}{2t^2}J^2\right. \\
&\quad + \frac{\sqrt{\alpha}e^{2\beta-2U}}{2t}\int_{R^3}\frac{f(1+2e^{-2U}v_2^2)}{|v_0|}dv_1\,dv_2\,dv_3\bigg\} \\
&\quad + \frac{e^{2U}}{2t}(A_t \pm \sqrt{\alpha}A_\theta)\left\{\frac{e^{2\beta-2U}}{2t^3}J(K - A\,J)\right. \\
&\quad + \frac{\sqrt{\alpha}e^{2\beta-2U}}{t^2}\int_{R^3}\frac{fv_2(v_3 - Av_2)}{|v_0|}dv_1\,dv_2\,dv_3\bigg\}. \tag{57}
\end{aligned}
$$

Let $u_1 = \sqrt{\alpha}v_1$.

$$
\begin{aligned}
(u_1^2)_s &= \frac{\alpha_t}{\alpha}u_1^2 + \frac{2\sqrt{\alpha}u_1}{v_0}\left\{e^{2\beta-2U}(\beta_\theta - U_\theta)\right. \\
&\quad + e^{2\beta-4U}(\beta_\theta - 2U_\theta)v_2^2 + \frac{e^{2\beta}}{t^2}(v_3 - Av_2)[(v_3 - Av_2)\beta_\theta \\
&\quad \left. - A_\theta v_2]\right\} + \frac{2e^{2\beta}u_1}{t}\left\{\frac{(K - A\,J)(v_3 - Av_2)}{t^2} + e^{-4U}Jv_2\right\} \tag{58}
\end{aligned}
$$

The advantage of $u_1$ over $v_1$ is that $\alpha_\theta$ does not appear in equation (58). Let $\bar{u}_1(t) = \sup\{\sqrt{\alpha}|v_1| : f(\hat{t}, \theta, v_1, v_2, v_3) \neq 0 \text{ for some } (\hat{t}, \theta, v_2, v_3) \in [t, t_i] \times S^1 \times R^2\}$. Analogously to the expanding direction in [4] and [7] (but here considering the contracting direction), the strategy to prove the proposition is to show that $\max_{S^1} E(t, \theta) + \bar{u}_1^2(t)$ is bounded on $(t_p, t_i]$. Let

$$
\Psi(t) = \begin{cases} \max_{S^1} E(t, \theta) + \bar{u}_1^2(t) & \text{if}\max_{S^1} E(t, \theta) + \bar{u}_1^2(t) > 1 \\ 1 & \text{if}\max_{S^1} E(t, \theta) + \bar{u}_1^2(t) \leq 1. \end{cases} \tag{59}
$$

If

$$
\Psi(t) \leq C_7 + \int_t^{t_i} h(\sigma)\Psi(\sigma)\ln\Psi(\sigma)\,d\sigma \tag{60}
$$

on $(t_p, t_i]$, for some number $C_7 > 1$ and function $h(t) \geq 0$, it follows (since $\Psi \geq 1$) that

$$
\frac{h(t)\Psi(t)\ln\Psi(t)}{(C_7 + \int_t^{t_i} h(\sigma)\Psi(\sigma)\ln\Psi(\sigma)\,d\sigma)\ln(C_7 + \int_t^{t_i} h(\sigma)\Psi(\sigma)\ln\Psi(\sigma)\,d\sigma)} \leq h(t).
$$

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Integrating both sides of this inequality and then exponentiating twice and using inequality (60) one finds that

$$\Psi(t) \le C_7^{\exp\left(\int_t^{t_i} h(\sigma)\,d\sigma\right)} \tag{61}$$

on $(t_p, t_i]$. It will be shown that $\Psi$ satisfies inequality (60) with the quantity $\int_t^{t_i} h(\sigma)\,d\sigma$ bounded on $(t_p, t_i]$, so that inequality (61) determines a bound for $\Psi$ on $(t_p, t_i]$.

First consider $\max_{S^1} E(t, \theta)$. Consider an arbitrary point $(t_k, \theta_k) \in (t_p, t_i] \times S^1$. Let $\gamma_\pm$ be the integral curves of $\partial_t \pm \sqrt{\alpha}\partial_\theta$ starting from the point $(t_k, \theta_k)$ and extending to the surface $t = t_i$. Let $(t_i, \theta_\pm)$ denote the endpoints of $\gamma_\pm$ lying on the surface $t = t_i$. Then

$$\int_{\gamma_-} (\partial_t - \sqrt{\alpha}\partial_\theta)(E + P) + \int_{\gamma_+} (\partial_t + \sqrt{\alpha}\partial_\theta)(E - P) = \int_{\gamma_-} L_+ + \int_{\gamma_+} L_-. \tag{62}$$

So

$$
\begin{aligned}
E(t_k, \theta_k) &= \frac{1}{2}\Big\{ E(t_i, \theta_+) - P(t_i, \theta_+) + E(t_i, \theta_-) + P(t_i, \theta_-) \\
&\qquad - \int_{\gamma_+} L_- - \int_{\gamma_-} L_+ \Big\}, \\
&\le E(t_i, \theta_+) + E(t_i, \theta_-) \\
&\qquad + \frac{1}{2}\left\{ \int_{\gamma_+} |L_-| + \int_{\gamma_-} |L_+| \right\}.
\end{aligned}
\tag{63}
$$

Consider $L_\pm$ (see equation (57)). The terms in the brackets satisfy

$$\frac{e^{2\beta - 4U}}{2t^2}J^2 + \frac{\sqrt{\alpha}e^{2\beta - 2U}}{2t}\int_{R^3} \frac{f(1 + 2e^{-2U}v_2^2)}{|v_0|}\,dv_1\,dv_2\,dv_3 \le |\frac{\alpha_t}{2\alpha t}|, \tag{64}$$

$$\frac{e^{2\beta - 2U}}{2t^3}J(K - A\,J) + \frac{\sqrt{\alpha}e^{2\beta - 2U}}{t^2}\int_{R^3} \frac{f v_2(v_3 - A v_2)}{|v_0|}\,dv_1\,dv_2\,dv_3 \le |\frac{\alpha_t}{2\alpha t}|. \tag{65}$$

The sum of the factors in front of the brackets satisfies

$$
\begin{aligned}
U_t \pm \sqrt{\alpha}U_\theta + \frac{e^{2U}}{2t}(A_t \pm \sqrt{\alpha}A_\theta) &\le (U_t \pm \sqrt{\alpha}U_\theta)^2 + \frac{e^{4U}}{4t^2}(A_t \pm \sqrt{\alpha}A_\theta)^2 + \frac{1}{2} \\
&\le 2(E \pm P) + \frac{1}{2} \\
&\le 4E + \frac{1}{2}.
\end{aligned}
\tag{66}
$$

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Thus

$$|L_{\pm}| \le |\frac{\alpha_t}{\alpha}|(2E + \frac{2E}{t} + \frac{1}{4t}) + \frac{3E}{t} \tag{67}$$

The first two terms in $\alpha_t/\alpha$ (see equation (15)) are treated in lemmas 14 and 15. Changing variables from $v_1$ to $u_1$, the third term satisfies

$$
\begin{aligned}
& 2e^{2\beta-2U} \int_{R^3} \frac{f(1 + e^{-2U}v_2^2 + t^{-2}e^{2U}(v_3 - Av_2)^2)}{|v_0|} du_1\, dv_2\, dv_3 \\
\le\ & 2e^{2\beta-2U} \int_{R^3} \frac{f(1 + e^{-2U}v_2^2 + t^{-2}e^{2U}(v_3 - Av_2)^2)}{\sqrt{e^{2\beta-2U} + u_1^2}} du_1\, dv_2\, dv_3 \tag{68} \\
\le\ & 8C_1(e^{\beta-U} + e^{\beta-3U}\bar{v}_2^2 + \frac{e^{\beta+U}}{t^2}(\bar{v}_3 + |A|\bar{v}_2)^2)\bar{v}_2\bar{v}_3 \int_{-\bar{u}_1}^{\bar{u}_1} \frac{du_1}{\sqrt{1 + e^{-2\beta+2U}u_1^2}} \\
\le\ & 16C_1\Big(e^{\beta-U} + e^{\beta-3U}\bar{v}_2^2 \\
& + \frac{e^{\beta+U}}{t^2}(\bar{v}_3 + |A|\bar{v}_2)^2\Big)\bar{v}_2\bar{v}_3\Big(e^{-1} + e^{\beta-U}\ln(\bar{u}_1 + \sqrt{e^{2\beta-2U} + \bar{u}_1^2})\Big). \tag{69}
\end{aligned}
$$

One of the factors, $e^{\beta-U}$, appearing in front of the integral in line (68) is used to control the terms appearing in the numerator and the other has been absorbed into the integral in the following line, to control it, since it has not yet been shown that $e^{\beta-U}$ is bounded away from zero. Now using lemmas 12 and 13, and inequalities (63), (67) and (69),

$$
\begin{aligned}
\max_{S^1} E(t_k, \theta) \le\ & 2\max_{S^1} E(t_i, \theta) \\
& + \int_{t_k}^{t_i} \Big\{\Big\{ \frac{\max_{S^1}[e^{2\beta(t,\theta)-4U(t,\theta)}(J(t,\theta))^2]}{t} \\
& + \frac{\max_{S^1}[e^{2\beta(t,\theta)}(K(t,\theta) - A(t,\theta)\,J(t,\theta))^2]}{t^3} \\
& + C_8(e^{-1} + e^{\beta-U}\ln(\max\{\bar{u}_1, e^{\beta-U}\}))\Big\}(2E + \frac{2E}{t_p} + \frac{1}{4t_p}) \\
& + \frac{3E}{t_p}\Big\}dt, \tag{70}
\end{aligned}
$$

for some positive number $C_8$.

Next consider $\bar{u}_1^2$. Integrating along geodesics using equation (58), treating $\alpha_t/\alpha$ as above, and using that $|u_1| < |v_0|$, that

$$\sqrt{\alpha}|\beta_\theta| \le 2tE + 4\bar{u}_1^2\bar{v}_2\bar{v}_3C_1, \tag{71}$$

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that $\sqrt{\alpha}|U_\theta| \le 2E + 1$ and that $\sqrt{\alpha}e^{2U}|A_\theta|/t \le 2E + 1$,

$$
\begin{aligned}
(u_1(t_k, \theta_k))^2 \le \ & (\bar{u}_1(t_i))^2 + \int_{t_k}^{t_i} \Bigg\{ \Bigg\{ \frac{\max_{S^1}[e^{2\beta(t,\theta)-4U(t,\theta)}(J(t,\theta))^2]}{t} \\
& + \frac{\max_{S^1}[e^{2\beta(t,\theta)}(K(t,\theta) - A(t,\theta)\,J(t,\theta))^2]}{t^3} \\
& + C_8(e^{-1} + e^{\beta-U}\ln(\max\{\bar{u}_1, e^{\beta-U}\})) \Bigg\}(\bar{u}_1(t))^2 \\
& + C_9(E + \bar{u}_1^2 + 1)\Bigg\{ \max_{S^1} e^{2\beta-2U} + (\max_{S^1} e^{2\beta-4U})\bar{v}_2^2 \\
& + \frac{[(\max_{S^1} e^\beta)\bar{v}_3 + (\max_{S^1} e^\beta|A|)\bar{v}_2]^2}{t_p^2} \\
& + \frac{[(\max_{S^1} e^{2\beta-2U})\bar{v}_3 + (\max_{S^1} e^{2\beta-2U}|A|)\bar{v}_2]\bar{v}_2}{t_p} \\
& + \frac{\max_{S^1}(e^\beta|K - A\,J|)[(\max_{S^1}(e^\beta)\bar{v}_3 + \max_{S^1}(e^\beta|A|)\bar{v}_2]}{t_p^3} \\
& + \frac{(\max_{S^1} e^{\beta-2U})\max_{S^1}(e^{\beta-2U}|J|)\bar{v}_2}{t} \Bigg\} \Bigg\}\,dt
\end{aligned}
\tag{72}
$$

for some positive number $C_9$. From (70) and (72), and using lemmas 14 and 15, it follows that $\Psi(t)$ satisfies inequality (60) with $\int_t^{t_i} h(\sigma)\,d\sigma$ bounded on $(t_p, t_i]$, so that inequality (61) determines a bound for $\Psi$ on $(t_p, t_i]$. That the support of $f$ and $U_t$ are bounded on $(t_p, t_i] \times S^1$ follows immediately. Therefore $U$ is bounded and from this follows the bound on $A_t$ and therefore $A$. The bounds on the derivatives with respect to $\theta$ follow from the bound on $E(t, \theta)$ and the fact that $\alpha$ is increasing with decreasing $t$. $\qquad\square$

**Corollary 2** $\ln \alpha$ is bounded on $(t_p, t_i] \times S^1$.

**Corollary 3** $J$, $K$ and their first derivatives are bounded on $(t_p, t_i] \times S^1$.

**Corollary 4** The functions $\alpha_t$, $\beta$, $\beta_t$, $\beta_\theta$, $\nu$ and $\nu_t$ are bounded on $(t_p, t_i] \times S^1$.

**Proposition 2** The first derivatives of $f$ are bounded on $(t_p, t_i] \times S^1 \times R^3$ and $\alpha_\theta$ is bounded on $(t_p, t_i] \times S^1$.

*Proof:* The appropriate quantities have now been bounded so as to allow this argument to proceed as for the expanding direction in [4] and [7]. The

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orthonormal frame used in [4] and [7] is such that the components of the velocity vector are

$$\hat{v}^0 = -e^{-\beta+U}v_0, \tag{73}$$

$$\hat{v}^1 = \sqrt{\alpha}e^{-\beta+U}v_1, \tag{74}$$

$$\hat{v}^2 = e^{-U}v_2, \tag{75}$$

$$\hat{v}^3 = \frac{e^U}{t}(v_3 - Av_2). \tag{76}$$

Note that $\hat{v}^0 = \sqrt{1 + (\hat{v}^1)^2 + (\hat{v}^2)^2 + (\hat{v}^3)^2}$ and that the support of $f$ is bounded in terms of these variables also. Let $Q = (s, \phi, w_1, w_2, w_3)$, $R = (t, \theta, v_1, v_2, v_3)$ and $S = (t, \theta, \hat{v}^1, \hat{v}^2, \hat{v}^3)$. Let $Q^0 = s$, $Q^1 = \phi$, $Q^{j+i} = w_i$ and analogously for coordinates $R$ and $S$. Let the indices $J$, $K$ and $L$ take on values from 0 to 4. The goal is to determine bounds for $\partial f/\partial R^J$.

The first step will be to determine bounds for

$$\frac{\partial f}{\partial S^J} = \frac{\partial f}{\partial Q^K}\frac{\partial Q^K}{\partial S^J}.$$

Since $\partial f/\partial Q^K$ is constant along timelike geodesics, for the first step it is enough to determine bounds for $\partial Q^K/\partial S^L$. Since the determinants of $\partial S/\partial R$ and $\partial R/\partial Q$ (see lemma 4) are bounded away from zero, the determinant of $\partial S/\partial Q$ is bounded away from zero, so it is enough to determine bounds for $\partial S^L/\partial Q^J$. Let the indices $a$, $b$, $c$ and $d$ take on values from 1 to 4. Since $\partial S^L/\partial Q^0$ (see equations (81) - (85) below) and $\partial S^0/\partial Q^J$ are bounded, it is enough to determine bounds for $\partial S^a/\partial Q^b$. Let

$$F_a^1 = \frac{1}{\sqrt{\alpha}}\frac{\partial\theta}{\partial Q^a}, \tag{77}$$

$$F_a^2 = \frac{\partial\hat{v}^1}{\partial Q^a} + \left\{\frac{\nu_t - U_t}{\sqrt{\alpha}}\hat{v}^0 + \frac{U_t}{\sqrt{\alpha}}\frac{(\hat{v}^3)^2 - (\hat{v}^2)^2}{\hat{v}^0}\right.$$
$$+ \left(\frac{U_t}{\sqrt{\alpha}}\frac{\hat{v}^1}{\hat{v}^0} - U_\theta\right)\hat{v}^1\frac{(\hat{v}^3)^2 - (\hat{v}^2)^2}{(\hat{v}^0)^2 - (\hat{v}^1)^2} - \frac{e^{2U}}{t}\frac{A_t}{\sqrt{\alpha}}\frac{\hat{v}^2\hat{v}^3}{\hat{v}^0}$$
$$\left.- \frac{e^{2U}}{t}\left(\frac{A_t}{\sqrt{\alpha}}\frac{\hat{v}^1}{\hat{v}^0} - A_\theta\right)\frac{\hat{v}^1\hat{v}^2\hat{v}^3}{(\hat{v}^0)^2 - (\hat{v}^1)^2}\right\}\frac{\partial\theta}{\partial Q^a}, \tag{78}$$

$$F_a^3 = \frac{\partial\hat{v}^2}{\partial Q^a} + U_\theta\hat{v}^2\frac{\partial\theta}{\partial Q^a}, \tag{79}$$

$$F_a^4 = \frac{\partial\hat{v}^3}{\partial Q^a} - (U_\theta\hat{v}^3 - \frac{e^{2U}}{t}A_\theta\hat{v}^2)\frac{\partial\theta}{\partial Q^a}. \tag{80}$$

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($F_b^a$ is the same as $(\Psi, Z^i)$ in [4, 7].) It is straightforward to see that $F_b^a = \Pi_c^a \partial S^c / \partial Q^b$, with $\Pi_c^a$ and its inverse bounded. Calculation of $\partial F_b^a / \partial s$, using

$$t_s = 1, \tag{81}$$

$$\theta_s = \frac{\sqrt{\alpha} \hat{v}^1}{\hat{v}^0}, \tag{82}$$

$$\hat{v}_s^1 = -(\beta_\theta - U_\theta)\sqrt{\alpha}\hat{v}^0 - (\beta_t - U_t - \frac{\alpha_t}{2\alpha})\hat{v}^1 - \sqrt{\alpha}U_\theta \frac{(\hat{v}^3)^2 - (\hat{v}^2)^2}{\hat{v}^0}$$
$$+ \frac{\sqrt{\alpha}e^{2U}}{t}A_\theta \frac{\hat{v}^2 \hat{v}^3}{\hat{v}^0} - \frac{e^{\beta-2U}}{t}J\hat{v}^2 - \frac{e^\beta}{t^3}(K - AJ)\hat{v}^3, \tag{83}$$

$$\hat{v}_s^2 = -U_t \hat{v}^2 - \sqrt{\alpha}U_\theta \frac{\hat{v}^1 \hat{v}^2}{\hat{v}^0}, \tag{84}$$

$$\hat{v}_s^3 = (U_t - \frac{1}{t})\hat{v}^3 + \sqrt{\alpha}U_\theta \frac{\hat{v}^1 \hat{v}^3}{\hat{v}^0} - \frac{e^{2U}}{t}(A_t \hat{v}^2 + \sqrt{\alpha}A_\theta \frac{\hat{v}^1 \hat{v}^2}{\hat{v}^0}), \tag{85}$$

shows that $\partial F_b^a / \partial s = \Lambda_c^a \partial S^c / \partial Q^b$, with $\Lambda_c^a$ bounded. (All terms appearing during the course of the calculation for which bounds have not yet been determined are canceled by other terms, with use of the evolution equations (20), (21) and (23).) Therefore $\partial F_b^a / \partial s = \Lambda_c^a (\Pi^{-1})_d^c F_b^d$ is sufficient to determine bounds for $F_b^a$, and therefore $\partial S^a / \partial Q^b$.

The second step is to determine a bound for $\alpha_\theta$. From the evolution equation (15),

$$\alpha_t = -\frac{\alpha e^{2\beta - 4U}J^2}{t} - \frac{\alpha e^{2\beta}(K - AJ)^2}{t^3}$$
$$-2\alpha t e^{2\beta-2U}\int_{R^3} \frac{f(1 + (\hat{v}^2)^2 + (\hat{v}^3)^2)}{\hat{v}^0}d\hat{v}^1\, d\hat{v}^2\, d\hat{v}^3. \tag{86}$$

Taking the derivative with respect to $\theta$ of both sides of this equation and then integrating in time leads to an inequality of the form

$$|\alpha_\theta| \leq C_{10} + \int_t^{t_i} |\alpha_\theta| \rho\, d\sigma, \tag{87}$$

with $C_{10} > 0$ constant and $\rho(t)$ nonnegative and bounded on $(t_p, t_i]$, so a bound for $\alpha_\theta$ is determined by Gronwall's lemma.

Finally,

$$\frac{\partial f}{\partial R^J} = \frac{\partial f}{\partial S^K}\frac{\partial S^K}{\partial R^J}. \tag{88}$$

determines bounds for $\partial f / \partial R^J$. $\qquad\square$

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**Corollary 5** $\nu_\theta$, $\alpha_{tt}$ and $\alpha_{t\theta}$ are bounded on $(t_p, t_i] \times S^1$.

**Lemma 16** The second derivatives of $U$ and $A$ are bounded on $(t_p, t_i] \times S^1$.

*Proof:* Let

$$E^1 = \frac{1}{2}\left\{(U_{tt})^2 + \alpha(U_{t\theta})^2 + \frac{e^{4U}}{4t^2}[(A_{tt})^2 + \alpha(A_{t\theta})^2]\right\}, \tag{89}$$

$$P^1 = \sqrt{\alpha}\left(U_{tt}U_{t\theta} + \frac{e^{4U}}{4t^2}A_{tt}A_{t\theta}\right). \tag{90}$$

A bound for $\max_{S^1} E^1(t, \theta)$ for $t \in (t_p, t_i]$ can be obtained using Gronwall's lemma by again considering an arbitrary point $(t_k, \theta_k) \in (t_p, t_i] \times S^1$, integrating $(\partial_t \mp \sqrt{\alpha}\partial_\theta)(E^1 \pm P^1)$ along $\gamma_\mp$ (as defined in the proof of proposition 1) and considering maxima on $S^1$ to obtain

$$\max_{S^1} E^1(t_k, \theta) \leq C_{11} + \int_{t_k}^{t_i} \max_{S^1} E^1(t, \theta)\, \xi \, dt, \tag{91}$$

for some positive number $C_{11}$ and some function $\xi(t)$ which is bounded and nonnegative on $(t_p, t_i]$. The bound for $\max_{S^1} E^1(t, \theta)$ on $(t_p, t_i]$, determines bounds for $U_{tt}$, $U_{t\theta}$, $A_{tt}$ and $A_{t\theta}$ on $(t_p, t_i] \times S^1$. The bounds for $U_{\theta\theta}$ and $A_{\theta\theta}$ then follow from equations (20) and (21). $\qquad\square$

**Corollary 6** The second derivatives of $J$, $K$ and $\beta$ are bounded on $(t_p, t_i] \times S^1$. $\nu_{t\theta}$ and $\nu_{tt}$ are bounded on $(t_p, t_i] \times S^1$.

**Proposition 3** Let $k$ be any positive integer. All $k$th order derivatives of the matter distribution function and of the metric functions are bounded on $(t_p, t_i] \times S^1 \times R^3$ and on $(t_p, t_i] \times S^1$, respectively.

*Proof:* This follows from an inductive argument, as follows. Let $n$ be any integer greater than one. Suppose that bounds have been determined for all derivatives of $\nu$, $\beta$, $\alpha$, $U$, $A$, $J$ and $K$ up to and including order $n$ except for $\partial^n\nu/\partial\theta^n$ and $\partial^n\alpha/\partial\theta^n$. In addition, suppose that bounds have been determined for $\partial S/\partial Q$ up to and including order $n - 1$. Let

$$F^a_{(n)b_1\cdots b_n} = \frac{\partial^n S^a}{\partial Q^{b_1}\cdots\partial Q^{b_n}} + \frac{1}{\sqrt{\alpha}}\frac{\partial^{n-1}\Gamma^a}{\partial\theta^{n-1}}\frac{\partial\theta}{\partial Q^{b_1}}\cdots\frac{\partial\theta}{\partial Q^{b_n}}, \tag{92}$$

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with

$$\Gamma^1 = -\sqrt{\alpha} \tag{93}$$

$$\Gamma^2 = (\nu_t - U_t)\hat{v}^0$$
$$+ U_t \frac{(\hat{v}^3)^2 - (\hat{v}^2)^2}{\hat{v}^0} + (U_t \frac{\hat{v}^1}{\hat{v}^0} - \sqrt{\alpha} U_\theta)\hat{v}^1 \frac{(\hat{v}^3)^2 - (\hat{v}^2)^2}{(\hat{v}^0)^2 - (\hat{v}^1)^2}$$
$$- \frac{e^{2U}}{t} \left[ A_t \frac{\hat{v}^2 \hat{v}^3}{\hat{v}^0} + \left( A_t \frac{\hat{v}^1}{\hat{v}^0} - \sqrt{\alpha} A_\theta \right) \frac{\hat{v}^1 \hat{v}^2 \hat{v}^3}{(\hat{v}^0)^2 - (\hat{v}^1)^2} \right] \tag{94}$$

$$\Gamma^3 = \sqrt{\alpha} U_\theta \hat{v}^2 \tag{95}$$

$$\Gamma^4 = -\sqrt{\alpha} U_\theta \hat{v}^3 + \frac{e^{2U}}{t} \sqrt{\alpha} A_\theta \hat{v}^2. \tag{96}$$

Similar to the proof of proposition 2, integrating $\partial_s F^a_{(n)b_1 \cdots b_n}$ along timelike geodesics determines bounds for $\frac{\partial^n S^a}{\partial Q^{b_1} \cdots \partial Q^{b_n}}$, followed by a bound for $\partial^n \alpha / \partial \theta^n$ and finally bounds for $\partial^n f / \partial R^{J_1} \cdots \partial R^{J_n}$. The evolution equations now determine bounds for $\partial^n \nu / \partial \theta^n$, and the $(n+1)$th order derivatives of $\alpha$, except for $\partial^{n+1} \alpha / \partial \theta^{n+1}$. Now bounds can be determined for the $(n+1)$th order derivatives of $U$ and $A$ by carrying out $n$ different light cone arguments, with

$$E^n_{mr} \pm P^n_{mr} = \frac{1}{2} \Bigg\{ \left( \frac{\partial^{n-1} U_{tt}}{\partial \theta^m \partial t^r} \pm \sqrt{\alpha} \frac{\partial^{n-1} U_{t\theta}}{\partial \theta^m \partial t^r} \right)^2$$
$$+ \frac{e^{4U}}{4t^2} \left( \frac{\partial^{n-1} A_{tt}}{\partial \theta^m \partial t^r} \pm \sqrt{\alpha} \frac{\partial^{n-1} A_{t\theta}}{\partial \theta^m \partial t^r} \right)^2 \Bigg\}, \tag{97}$$

for each choice of nonnegative integers $m$ and $r$ such that $m + r = n - 1$. Finally, the evolution equations determine bounds for all $(n+1)$th order derivatives of $J$, $K$ and $\beta$, and $\nu$ except for $\partial^{n+1} \nu / \partial \theta^{n+1}$. $\qquad \square$

## 5 Conclusion

**Theorem 1** *The area of the $T^2$ group orbits takes on all positive values on the maximal Cauchy development of $T^2$ symmetric initial data for the Einstein-Vlasov equations on $T^3$ with a positive number of Vlasov particles on the initial data surface, all of the same nonzero mass.*

*Proof:* Suppose that $t_p = t_0$. From the bound on the support of the matter distribution function and the $C^\infty$ bounds obtained in section 4, it follows that

there is a $C^\infty$ extension of the metric functions and the matter distribution function to $t_p$ (satisfying the Einstein-Vlasov equations). Therefore $\{t_p\} \times S^1$ is contained in the Cauchy development of any Cauchy surface in $(t_p, \infty) \times S^1$. This contradicts that the Cauchy development does not extend to $\{t_0\} \times S^1$. Since $t_p$ was allowed to be any positive number in $[t_0, \infty)$, it must be the case that $t_0 = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In [7] it is shown that there is a crushing singularity as $t \downarrow t_0$ and that there exists a foliation of the maximal Cauchy development by compact spatial hypersurfaces of constant mean curvature.

# 6   Acknowledgments

# 7   Appendix

An argument by Chruściel concerning vacuum $T^2$ symmetric spacetimes on $T^3$ (see pp 113-114 of [2]) carries over to Einstein-Vlasov $T^2$ symmetric spacetimes on $T^3$ to show that the presence of Vlasov matter guarantees that the spacetime gradient of the area of the $T^2$ symmetry orbits is everywhere non-vanishing. In this appendix the argument is recalled. The areal form (1) of the metric is only valid if the spacetime gradient of the area of the $T^2$ symmetry orbits, $t$, is nonvanishing. However, the conformal form of the metric,

$$
\begin{aligned}
g \;=\; & e^{2(\tilde{\nu}-U)}(-d\tilde{t}^2 + d\tilde{\theta}^2) + e^{2U}[d\tilde{x} + A\,d\tilde{y} + (\tilde{G} + A\tilde{H})\,d\tilde{\theta}]^2 \\
& + e^{-2U}t^2[d\tilde{y} + \tilde{H}\,d\tilde{\theta}]^2,
\end{aligned} \tag{98}
$$

in terms of conformal coordinates $(\tilde{t}, \tilde{\theta}, \tilde{x}, \tilde{y})$, is valid even if the gradient of $t$ vanishes. (Given a spacetime with metric (1), one can set $\partial_x = \partial_{\tilde{x}}$ and $\partial_y = \partial_{\tilde{y}}$, but to insure zero shift it may be that $x \neq \tilde{x}$ and $y \neq \tilde{y}$.) The twist quantities are related to the conformal metric functions by

$$
J = -t e^{-2\tilde{\nu}+4U}(\tilde{G}_{\tilde{t}} + A\tilde{H}_{\tilde{t}}) \quad \text{and} \quad K = A\,J - t^3 e^{-2\tilde{\nu}}\tilde{H}_{\tilde{t}}. \tag{99}
$$

Let the frame for the velocity of the particles be

$$\{d\tilde{t}, \ d\tilde{\theta}, \ d\tilde{x} + \tilde{G}\, d\tilde{\theta}, \ d\tilde{y} + \tilde{H}\, d\tilde{\theta}\}. \tag{100}$$

Then

$$\tilde{v}_0 = -\sqrt{e^{2\tilde{\nu}-2U} + \tilde{v}_1^2 + e^{2\tilde{\nu}-4U}\tilde{v}_2^2 + t^{-2}e^{2\tilde{\nu}}(\tilde{v}_3 - A\tilde{v}_2)^2}. \tag{101}$$

Let $h_+ = h_{\tilde{t}} + h_{\tilde{\theta}}$ and $h_- = h_{\tilde{t}} - h_{\tilde{\theta}}$ for any function, $h$. Let

$$\begin{aligned}
B &= t(U_+^2 + \frac{e^{4U}}{4t^2}A_+^2) + \frac{e^{2\tilde{\nu}-4U}}{4t}J^2 \\
&\quad + \frac{e^{2\tilde{\nu}}}{4t^3}(K - A\,J)^2 + \int_{R^3} f(|\tilde{v}_0| - \tilde{v}_1)\, d\tilde{v}_1\, d\tilde{v}_2\, d\tilde{v}_3, \tag{102} \\
D &= t(U_-^2 + \frac{e^{4U}}{4t^2}A_-^2) + \frac{e^{2\tilde{\nu}-4U}}{4t}J^2 \\
&\quad + \frac{e^{2\tilde{\nu}}}{4t^3}(K - A\,J)^2 + \int_{R^3} f(|\tilde{v}_0| + \tilde{v}_1)\, d\tilde{v}_1\, d\tilde{v}_2\, d\tilde{v}_3. \tag{103}
\end{aligned}$$

Note that $B$ and $D$ are both strictly positive since $|\tilde{v}_1| < |\tilde{v}_0|$ and since, by assumption, the number of Vlasov particles present is nonvanishing (see lemma 2 and the paragraph following its proof). The following equations can be derived from equations (5) and (6) in [7].

$$\begin{aligned}
(t_+)_{\tilde{\theta}} &= -B + \tilde{\nu}_+ t_+, \tag{104} \\
(t_-)_{\tilde{\theta}} &= D - \tilde{\nu}_- t_-. \tag{105}
\end{aligned}$$

So on any $\tilde{t} = $ constant surface,

$$t_+(\tilde{t}, \tilde{\theta}) = t_+(\tilde{t}, \tilde{\theta}_0)e^{\int_{\tilde{\theta}_0}^{\tilde{\theta}} \tilde{\nu}_+(\tilde{t},\lambda)\, d\lambda} - \int_{\tilde{\theta}_0}^{\tilde{\theta}} B(\tilde{t}, \phi)e^{\int_{\phi}^{\tilde{\theta}} \tilde{\nu}_+(\tilde{t},\lambda)\, d\lambda}d\phi \tag{106}$$

$$t_-(\tilde{t}, \tilde{\theta}) = t_-(\tilde{t}, \tilde{\theta}_0)e^{-\int_{\tilde{\theta}_0}^{\tilde{\theta}} \tilde{\nu}_-(\tilde{t},\lambda)\, d\lambda} + \int_{\tilde{\theta}_0}^{\tilde{\theta}} D(\tilde{t}, \phi)e^{-\int_{\phi}^{\tilde{\theta}} \tilde{\nu}_-(\tilde{t},\lambda)\, d\lambda}d\phi \tag{107}$$

Thus, if $t_+(\tilde{t}, \tilde{\theta}_0) = 0$ for some $(\tilde{t}, \tilde{\theta}_0)$, then $t_+(\tilde{t}, \tilde{\theta}) < 0$ for $\tilde{\theta} > \tilde{\theta}_0$. And if $t_-(\tilde{t}, \tilde{\theta}_0) = 0$ for some $(\tilde{t}, \tilde{\theta}_0)$, then $t_-(\tilde{t}, \tilde{\theta}) > 0$ for $\tilde{\theta} > \tilde{\theta}_0$. This is not compatible with the periodicity of $t_\pm$ in $\tilde{\theta}$, so $t_\pm(\tilde{t}, \tilde{\theta}_0) = 0$ is not possible. (In vacuum and if $J = 0$ and $K = 0$, periodicity implies that if $t_\pm(\tilde{t}, \tilde{\theta}_0) = 0$ for some $(\tilde{t}, \tilde{\theta}_0)$, then $t_\pm = 0$ on the whole $\tilde{t} = $ constant surface, which in turn implies that $t_\pm = 0$ in the maximal Cauchy development of the $\tilde{t} = $ constant surface, and that the spacetime is flat.)

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